Creep Stress Concentration at a Circular Hole in an Infinite Plate

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The present paper is concerned with the creep problem of stress concentration at holes in thin sheets subjected to loads in their own plane. In particular, the problem of creep stress concentration at a small circular hole in an infinite thin plate subjected to uniaxial tension is analyzed in the first part of the paper. The second part is essentially an extension of the preceding analysis to the case when the loading is biaxial and consists of tension and compression. These analyses are based on a creep power law and an analogy between the problem of creep and the corresponding nonlinear elastic problem. The solution of this problem is based on Galerkin's method. The stress solutions for some values of the index of the power law are presented.

1. Introduction

CONSIDERABLE effort has been directed toward developing mathematical analyses of creep based upon the type of creep relations proposed by experimentalists. In principle, this is feasible since the creep analysis of any structure consists of satisfying the conditions of equilibrium and compatibility in addition to the assumed stress-strain rate relation. Thus, it is often possible to derive the differential equation that governs the distribution of stress and displacement in a particular structure. However, at variance with the situation in the classical theory of elasticity, the derived partial differential equations are in the majority of cases highly nonlinear, and their direct solution is extremely difficult. The success in obtaining solutions by this procedure has been limited, in general, to problems that have radial symmetry or point symmetry. In these cases the simplicity of having to deal with a single independent space variable often leads to ordinary differential equations either in the stresses or in the displacements. Numerous standard techniques exist for the solution of such equations. Even so, situations occur where direct solutions become mathematically intractable. In such cases, alternative methods of solution have to be found. This necessity becomes almost imperative in problems lacking symmetry.

The present paper is concerned with the creep problem of stress concentration at holes in thin sheets subjected to loads in their own plane. In particular, the problem of creep stress concentration at a small circular hole in an infinite thin plate subjected to uniaxial tension is analyzed in the first part of the paper. The second part is essentially an extension of the preceding analysis to the case when the loading is biaxial and consists of tension and compression. These analyses are based on a creep power law. It is assumed that the material of the plates is homogeneous, isotropic, and incompressible. Further, it is assumed that thermal stresses do not arise. In addition, instantaneous elastic deformations are considered negligible in comparison with creep deformations. The analyses are based on an analogy between the problem of creep and the corresponding nonlinear elastic problem. Thus, the methods of collocation and Galerkin are first used in the solution of the governing equation $\nabla^4\varphi = 0$ of the linear elastic problem. Although both methods of solution yield results in good agreement with the exact one, the use of the collocation technique proves mathematically less tractable than the use of the Galerkin method. Moreover, the application of the former technique to the nonlinear problem results in nonlinear algebraic equations whose solution by iterative procedures proves prohibitively difficult. Hence, Galerkin's method is adopted for the solution of the nonlinear problems under consideration. Details of the reasons that led to the use of the latter method are included in the body of the paper.

Before proceeding with the actual analyses of the problems outlined, some comments on the reasons that led to the present investigations are deemed appropriate. An earlier investigation of the problem of creep-stress concentration at a circular hole in a plate subjected to simple tension was made by Marin.[1] He considered a plate of finite width $2b$ with a central circular hole of radius a, subjected to a uniaxial tension in the longitudinal direction. Since the problem under consideration lacks point symmetry and hence is difficult to solve directly, Marin proposed an approximate method of determining the creep-stress distribution across the lateral widths $b - a$ of the plate by considering a ring of inner radius a and outer radius b. In such a case, the loading on the ring would be symmetric about the longitudinal and lateral axes passing through the center of the hole, and hence only a quadrant of the ring need be analyzed. Subject to these considerations, Marin developed an engineering analysis by investigating the quadrant as a curved bar fixed at the end in the longitudinal direction. The loading of the bar consisted of varying longitudinal forces at its outer boundary, and moment and force reactions applied at its lateral cross section. The present approach differs radically from that of Marin. Here, the problem is considered as one in elasticity by virtue of the elastic analog, and the governing equations are satisfied in approximation over the entire plate.

2. Creep Law and Its Analog

There exist different formulations of laws relating creep deformation to its various governing parameters. Based on Prager's[2] general stress-strain law for incompressible iso-

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tropic elastic material, Hoff[3] proposed a comparatively simple law to account for creep in structural materials subject to a triaxial state of stress. This law may be written

$$\epsilon_{ij} = CJ_2{}^m s_{ij} t^{1/p} \tag{2.1}$$

where t represents time, ϵ_{ij} is the strain tensor, s_{ij} is the stress deviation tensor defined by

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} \tag{2.2}$$

and J_2 is the second invariant of the stress deviation tensor

$$J_2 = \tfrac{1}{2}s_{ij}s_{ij} \tag{2.3}$$

The material parameters C, m, and p in Eq. (2.1) must be obtained from constant temperature creep tests.

Hoff[4] has shown that the creep problem of a structure governed by Eq. (2.1) may be transformed to an analogous nonlinear elastic problem. If it is assumed that the elastic problem is based on the nonlinear law

$$\epsilon_{ij} = CJ_2{}^m s_{ij} \tag{2.4}$$

and the relation between the elastic and creep strains is

$$(\epsilon_{ij})_{cr} = \epsilon_{ij} t^{1/p} \tag{2.5}$$

then the two problems are mathematically identical. Thus, the same structure subject to the same loads is to be considered, but the structure is now to be governed by Eq. (2.4) and the velocities are to be obtained from the relation

$$u^p = (d/dt)[u_{cr}{}^p] \tag{2.6}$$

Here, u and u_{cr} represent, respectively, the elastic and creep displacements. The succeeding analyses in the present paper are based on Eq. (2.4).

3. Derivation of the Governing Equation

The problems under consideration are illustrated in Fig. 1. Since the loads are applied in the plane of the plates, both problems fall in the category of plane stress. Thus, in polar coordinates, the only nonzero stress components are the radial stress σ_r, the circumferential stress σ_θ, and the shearing stress $\tau_{r\theta}$. These stresses are governed by the equilibrium equations

$$\begin{gathered}(\partial\sigma_r/\partial r) + (1/r)(\partial\tau_{r\theta}/\partial\theta) + [(\sigma_r - \sigma_\theta)/r] = 0\\ (1/r)(\partial\sigma_\theta/\partial\theta) + (\partial\tau_{r\theta}/\partial r) + (2\tau_{r\theta}/r) = 0\end{gathered} \tag{3.1}$$

Further, assuming small deformations, the only compatibility equation that the strains must satisfy is

$$\frac{\partial^2\epsilon_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\epsilon_r}{\partial\theta^2} + \frac{2}{r}\frac{\partial\epsilon_\theta}{\partial r} - \frac{1}{r}\frac{\partial\epsilon_r}{\partial r} = \frac{1}{r}\frac{\partial^2\gamma_{r\theta}}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\gamma_{r\theta}}{\partial\theta} \tag{3.2}$$

Here, the strains ϵ_r, ϵ_θ, and $\gamma_{r\theta}$ are the radial strain, the circumferential strain, and shearing strain, respectively. The nonzero axial strain $\epsilon_z = -(\epsilon_r + \epsilon_\theta)$ does not appear in the analysis of the problem. Finally, the material of the plates must satisfy the nonlinear stress-strain relation given by Eq. (2.4).

Subject to the preceding discussions, the stress deviations of the problem and their second invariant are

$$\begin{gathered}s_r = \tfrac{1}{3}(2\sigma_r - \sigma_\theta) \qquad s_\theta = \tfrac{1}{3}(2\sigma_\theta - \sigma_r)\\ s_z = -\tfrac{1}{3}(\sigma_r + \sigma_\theta) \qquad s_{r\theta} = \tau_{r\theta}\\ J_2 = \tfrac{1}{3}(\sigma_r{}^2 - \sigma_r\sigma_\theta + \sigma_\theta{}^2 + 3\tau_{r\theta}{}^2)\end{gathered} \tag{3.3}$$

It follows from Eq. (2.4) that the corresponding stress-strain relations are

$$\begin{gathered}\epsilon_r = (C/3)J_2{}^m(2\sigma_r - \sigma_\theta) \qquad \epsilon_\theta = (C/3)J_2{}^m(2\sigma_\theta - \sigma_r)\\ \gamma_{r\theta} = 2\epsilon_{r\theta} = 2C\,J_2{}^m\tau_{r\theta}\end{gathered} \tag{3.4}$$

Therefore, substitution for the strains in terms of the stresses in Eq. (3.2) from Eq. (3.4) leads to

$$\begin{aligned}&J_2{}^m\left\{\frac{\partial^2}{\partial r^2}(2\sigma_\theta - \sigma_r) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(2\sigma_r - \sigma_\theta) + \frac{1}{r}\frac{\partial}{\partial r}(5\sigma_\theta - 4\sigma_r) - \frac{6}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial\tau_{r\theta}}{\partial\theta}\right)\right\} + mJ_2{}^{m-1}\times\\ &\left\{\left(\frac{\partial^2 J_2}{\partial r^2}\right)(2\sigma_\theta - \sigma_r) + \left(\frac{\partial J_2}{\partial r}\right)\left[2\frac{\partial}{\partial r}(2\sigma_\theta - \sigma_r) + \frac{1}{r}(5\sigma_\theta - 4\sigma_r) - \frac{6}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}\right] + \frac{1}{r^2}\left(\frac{\partial^2 J_2}{\partial\theta^2}\right)(2\sigma_r - \sigma_\theta) + \right.\\ &\left.\left(\frac{\partial J_2}{\partial\theta}\right)\left[\frac{2}{r^2}\frac{\partial}{\partial\theta}(2\sigma_r - \sigma_\theta) - \frac{6}{r^2}\frac{\partial}{\partial r}(r\tau_{r\theta})\right] - \frac{6}{r}\tau_{r\theta}\frac{\partial^2 J_2}{\partial r\partial\theta}\right\} + m(m-1)J_2{}^{m-2}\left\{\left(\frac{\partial J_2}{\partial r}\right)^2(2\sigma_\theta - \sigma_r) + \right.\\ &\left.\left(\frac{1}{r}\frac{\partial J_2}{\partial\theta}\right)^2(2\sigma_r - \sigma_\theta) - \frac{6}{r}\tau_{r\theta}\frac{\partial J_2}{\partial r}\frac{\partial J_2}{\partial\theta}\right\} = 0\end{aligned} \tag{3.5}$$

This stress-compatibility equation together with Eqs. (3.1) represents a system of three equations in the three unknowns σ_r, σ_θ, and $\tau_{r\theta}$. The procedure for solving these equations is similar to that used in the linear theory of elasticity. Thus, if it is assumed that the stresses are of the form

$$\begin{gathered}\sigma_r = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} \qquad \sigma_\theta = \frac{\partial^2\varphi}{\partial r^2}\\ \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)\end{gathered} \tag{3.6}$$

where φ is the Airy stress function, the equilibrium equations [Eqs. (3.1)] are identically satisfied. Therefore, substitution of Eqs. (3.6) into (3.5) yields

$$\begin{aligned}&2J_2{}^m\nabla^4\varphi + mJ_2{}^{m-1}\left[J''_2\left(2\varphi'' - \frac{1}{r}\varphi' - \frac{1}{r^2}\ddot{\varphi}\right) + J'_2\left(4\varphi''' + \frac{3}{r}\varphi'' - \frac{2}{r^2}\varphi' - \frac{6}{r^3}\ddot{\varphi} + \frac{4}{r^2}\ddot{\varphi}'\right) + \right.\\ &\frac{\ddot{J}_2}{r^2}\left(\frac{2}{r}\varphi' + \frac{2}{r^2}\ddot{\varphi} - \varphi''\right) + \frac{\dot{J}_2}{r^2}\left(\frac{4}{r^2}\dddot{\varphi} + \frac{6}{r^2}\dot{\varphi} - \frac{2}{r}\dot{\varphi}' + 4\dot{\varphi}''\right) + \left.\frac{6}{r}\dot{J}'_2\left(\frac{\dot{\varphi}'}{r} - \frac{\dot{\varphi}}{r^2}\right)\right] + \\ &m(m-1)J_2{}^{m-2}\left[(J'_2)^2\left(2\varphi'' - \frac{1}{r}\varphi' - \frac{1}{r^2}\ddot{\varphi}\right) + \left(\frac{\dot{J}_2}{r}\right)^2\left(\frac{2}{r}\varphi' + \frac{2}{r^2}\ddot{\varphi} - \varphi''\right) + \frac{6}{r}J'_2\dot{J}_2\left(\frac{\dot{\varphi}'}{r} - \frac{\dot{\varphi}}{r^2}\right)\right] = 0\end{aligned} \tag{3.7}$$

where $(\ )' = \partial/\partial r$ and $(\dot{\ }) = \partial/\partial\theta$. Equation (3.7) is the governing equation in terms of φ for both problems under consideration. If $m = 0$, the equation reduces to the well-known biharmonic equation of the classical elasticity theory.

4. Plate Under Uniaxial Tension

If the applied tension σ_0 and the hole radius a (Fig. 1a) are taken as unities, Eq. (3.7) governing the problem can be treated as a dimensionless equation. Further, the boundary conditions of the problem are

$$\begin{gathered}\sigma_r = 0 \qquad \tau_{r\theta} = 0 \qquad \text{at } r = 1\\ \sigma_r = \tfrac{1}{2}(1 + \cos 2\theta) \qquad \sigma_\theta = \tfrac{1}{2}(1 - \cos 2\theta)\\ \tau_{r\theta} = -\tfrac{1}{2}\sin 2\theta \qquad \text{at } r = \infty\end{gathered} \tag{4.1}$$

It is readily seen that for $m > 0$ any attempt to solve directly

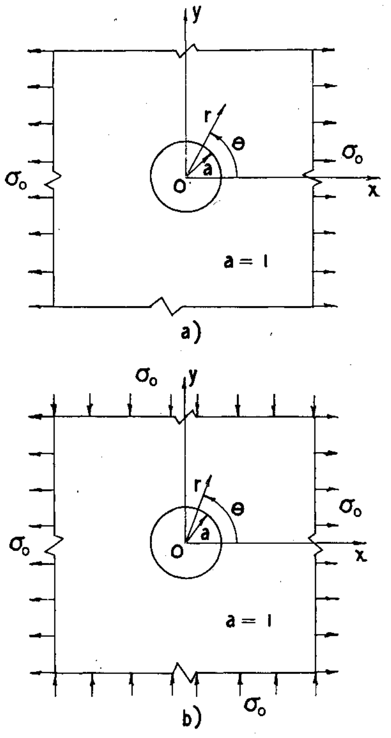

Fig. 1 Plates with circular holes.

Eq. (3.7) subject to the boundary conditions (4.1) would be extremely difficult. However, for $m = 0$ the exact solution is well known from the linear theory of elasticity. This solution provides a ready basis for testing out approximate procedures before they are applied to the case where $m > 0$. To this end, the methods of collocation and Galerkin are initially used in solving the equation $\nabla^4\varphi = 0$ subject to the boundary conditions (4.1). Although the former technique rests upon satisfying the differential equation at discrete points in the plate, the latter depends upon satisfying the integral of the differential equation over the entire plate.

As a preliminary to the use of either technique, it is first assumed that the function φ is of the series form

$$\varphi = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} A_{kl} r^{-k} \cos l\theta \tag{4.2}$$

Hence by virtue of Eqs. (3.6), the stresses assume the form

$$\sigma_r = -\sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} (k + l^2) A_{kl} r^{-(k+2)} \cos l\theta$$

$$\tau_{r\theta} = -\sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} l(k + 1) A_{kl} r^{-(k+2)} \sin l\theta \tag{4.3}$$

$$\sigma_\theta = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} k(k + 1) A_{kl} r^{-(k+2)} \cos l\theta$$

If the stresses are to remain finite and satisfy the boundary conditions (4.1), it follows that

$$k \geq -2 \qquad l = 0,2 \tag{4.4a}$$

$$\left.\begin{aligned}
\sigma_r = 0 &= -\sum_{k=-2}^{\infty} kA_{k0} - \sum_{k=-2}^{\infty} (k+4)A_{k2}\cos 2\theta \\
\tau_{r\theta} = 0 &= -\sum_{k=-2}^{\infty} (k+1)A_{k2}\sin 2\theta \\
\sigma_r &= \tfrac{1}{2}(1 + \cos 2\theta) = 2A_{-20} - 2A_{-22}\cos 2\theta \\
\tau_{r\theta} &= -\tfrac{1}{2}\sin 2\theta = 2A_{-22}\sin 2\theta \\
\sigma_\theta &= \tfrac{1}{2}(1 - \cos 2\theta) = 2A_{-20} + 2A_{-22}\cos 2\theta
\end{aligned}\right\} \tag{4.4b}$$

Since Eqs. (4.4b) must hold for all values of θ, the coefficients must satisfy the conditions

$$\sum_{k=-2}^{\infty} kA_{k0} = 0 \qquad \sum_{k=-2}^{\infty} (k+4)A_{k2} = 0$$
$$\sum_{k=-2}^{\infty} (k+1)A_{k2} = 0 \qquad A_{-20} = \frac{1}{4} \qquad A_{-22} = -\frac{1}{4} \tag{4.5}$$

Since A_{00} has no effect on the stresses, it is set equal to zero. Finally, by the use of either of the proposed techniques, the remaining coefficients must be determined by the requirement that the stress function φ given by Eq. (4.2) satisfies the equation $\nabla^4\varphi = 0$.

The evaluation of the remaining coefficients by the use of the collocation technique is considered first. Briefly, it is assumed that the plate is subdivided into grids made by the intersection of radial lines with circles that are concentric to the hole. The function φ given by Eq. (4.2) and subject to Eqs. (4.4a) and (4.5) is then made to satisfy $\nabla^4\varphi = 0$ at every mesh point of the grid. This procedure leads to the following set of linear algebraic equations:

$$\sum_{k=1}^{\infty} k[1 + k(k+2)^2 r^{-(k+1)}]A_{k0} = \frac{1}{2}$$
$$\sum_{k=1}^{\infty} k[9 + (k+4)(k^2 - 4)r^{-(k+1)}]A_{k2} = -\frac{9}{2} \tag{4.6}$$

In order that each set be a determinate one for the evaluation of the coefficients A_{k0} and A_{k2}, the number of terms that can be retained in each of the infinite series must be the same as the number of mesh points along a radial line. Evaluation of A_{k0} and A_{k2} from Eq. (4.6) for various values of the outer radius r and interval Δr and their substitution in Eq. (4.2) subject to Eqs. (4.4a) and (4.5) yield the stress distribution in the plate. This solution can be used to determine the circumferential stress σ_θ at $r = 1$ and $\theta = \pi/2$, the point of stress concentration. The results are summarized in Table 1.

The evaluation of the coefficiencies A_{k0} and A_{k2} by the use of Galerkin's method is considered next. With reference to the problem under consideration, the first step in this procedure is to assume a number of functions φ, each satisfying all the boundary conditions of the problem. As a rule, these functions will not satisfy the differential equation $\nabla^4\varphi = 0$ and will give rise to an error. The object is to make this error as small as possible, so that the assumed functions can be considered to be the solution of $\nabla^4\varphi = 0$ in sufficiently good approximation. To this end, it is assumed that the function φ is again of the series form shown in Eq. (4.2) and satisfies the boundary conditions (4.1). This means that the series is subject to the conditions on the coefficients given by Eqs. (4.4a) and (4.5). By Galerkin's method, the assumed function will be the solution of $\nabla^4\varphi = 0$ if it satisfies the condition

$$\int_0^{\pi/2}\int_1^{\infty} \nabla^4 \delta\varphi r dr d\theta = 0 \tag{4.7}$$

Further, if the variation $\delta\varphi$ is carried out in terms of each of the remaining coefficients in the series, Eq. (4.7) reduces to

Table 1 Stress concentration factor for the linear case by collocation method

No. of points	Δr			
	1	0.5	0.25	0.125
5	2.88071	2.93941	2.90804	2.88032
10	7.49931	3.03093	2.97841	2.94984
20	7.21984	11.37878	1.09942	2.96965

the two sets

$$\int_0^{\pi/2}\int_1^{\infty} \nabla^4\varphi\delta\varphi_{A_{l0}} r dr d\theta = 0 \quad (4.8)$$

$$\int_0^{\pi/2}\int_1^{\infty} \nabla^4\varphi\delta\varphi_{A_{l2}} r dr d\theta = 0 \quad (4.9)$$

where l takes on, in turn, each and every possible value of k. If the variations are carried out with respect to each of the infinite number of coefficients, then the solution of the corresponding infinite number of equations is the exact solution of $\nabla^4\varphi = 0$. Since in practice only a finite number of coefficients can be considered, the procedure would lead to an approximate solution of the latter equation.

Before proceeding with the aforementioned variations, it can be easily shown that the coefficients must satisfy other conditions in addition to Eqs. (4.4a) and (4.5). Examination of function φ given by Eq. (4.2) shows that for the Galerkin integrals [Eqs. (4.8) and (4.9)] to remain finite

$$A_{-10} = 0 \qquad A_{-12} = 0 \quad (4.10)$$

Enforcement of the conditions contained in Eqs. (4.4a, 4.5, and 4.10) on Eqs. (4.2) and (4.3) reduce the stress function and the stresses to the form

$$\varphi = \frac{1}{4}r^2 + \frac{1}{2}r^{-1} + \sum_{k=2}^{\infty}(-kr^{-1} + r^{-k})A_{k0} + \left\{-\frac{1}{4}r^2 + \frac{3}{4} - \frac{1}{2}r^{-1} + \sum_{k=2}^{\infty}[(k-1) - kr^{-1} + r^{-k}]A_{k2}\right\}\cos 2\theta$$

$$\sigma_r = f_1(r) + g_1(r)\cos 2\theta \quad (4.11a)$$

$$\sigma_\theta = f_2(r) + g_2(r)\cos 2\theta \qquad \tau_{r\theta} = g_3(r)\sin 2\theta$$

where

$$f_1(r) = \frac{1}{2} - \frac{1}{2}r^{-3} + \sum_{k=2}^{\infty} k[r^{-3} - r^{-(k+2)}]A_{k0}$$

$$f_2(r) = \frac{1}{2} + r^{-3} + \sum_{k=2}^{\infty} k[-2r^{-3} + (k+1)r^{-(k+2)}]A_{k0}$$

$$g_1(r) = \frac{1}{2} - 3r^{-2} + \frac{5}{2}r^{-3} + \sum_{k=2}^{\infty}[-4(k-1)r^{-2} + 5kr^{-3} - (k+4)r^{-(k+2)}]A_{k2} \quad (4.11b)$$

$$g_2(r) = -\frac{1}{2} - r^{-3} + \sum_{k=2}^{\infty} k[-2r^{-3} + (k+1)r^{-(k+2)}]A_{k2}$$

$$g_3(r) = -\frac{1}{2} - \frac{3}{2}r^{-2} + 2r^{-3} + \sum_{k=2}^{\infty}[-2(k-1)r^{-2} + 4kr^{-3} - 2(k+1)r^{-(k+2)}]A_{k2}$$

It follows that

$$\delta\varphi_{A_{l0}} = (\partial\varphi/\partial A_{l0}) = -lr^{-1} + r^{-l}$$

$$\delta\varphi_{A_{l2}} = (\partial\varphi/\partial A_{l2}) = [(l-1) - lr^{-1} + r^{-l}]\cos 2\theta \quad (4.12)$$

Substituting for φ from the first of Eqs. (4.11a), and for the variations of φ from Eqs. (4.12) into Eqs. (4.8) and (4.9) and succeeding integration, leads to the following set of uncoupled linear algebraic equations for the evaluation of A_{k0} and A_{k2};

$$\frac{9}{2}\left(-\frac{1}{l+3} + \frac{l}{4}\right) + \sum_{k=2}^{\infty} kA_{k0}\left[k(k+2)^2 \times \left(-\frac{1}{k+l+2} + \frac{l}{k+3}\right) - 9\left(-\frac{1}{l+3} + \frac{l}{4}\right)\right] = 0 \quad (4.13)$$

Table 2 Stress concentration factor for the linear case by Galerkin's method

No. of coeffs. in each set	Stress concentration factor
1	3.21678
2	3.04846
3	3.03503
5	3.03235
10	3.00946
20	3.00786

and

$$\frac{15}{2}\left[-\frac{(l-1)}{3} + \frac{l}{4} - \frac{1}{l+3}\right] + \sum_{k=2}^{\infty} kA_{k2} \times \left\{(l-1)\left[-5 - \frac{(k+4)(k^2-4)}{k+2}\right] + l\left[\frac{15}{4} + \frac{(k+4)(k^2-4)}{k+3}\right] - \left[\frac{15}{l+3} + \frac{(k+4)(k^2-4)}{k+l+2}\right]\right\} = 0$$

where $l = 2, 3, \ldots, \infty$. For a solution of these sets of equations, the number of values that l can take on is the same as the number of coefficients proposed for retention in the series. Thus, each value of l will correspond to one equation in either set and both sets will be determinate. These equations were solved for various choices of the number of coefficients. These coefficients were then used in Eq. (4.11a) to determine the circumferential stress σ_θ at $r = 1$ and $\theta = \pi/2$, the point of stress concentration. The results are summarized in Table 2.

As stated earlier, the reason for the preceding investigation was to determine the suitability of either technique in the solution of the nonlinear problem $m > 0$. With regards to the use of the collocation method, it can be seen from Table 1 that for larger values of the interval Δr, e.g., 1 and 0.5, inclusion of a larger portion of the plate in the analysis (that is, considering larger values of the outer radius) does not improve the accuracy of the solution. In fact, the solution becomes far less accurate. The only conclusion that can be drawn is that as more and more portions of the plate are included in the analysis with the intervals Δr being made less and less, the accuracy of the solution improves. As regards the use of Galerkin's technique, Table 2 shows that the retention of only one coefficient in each set results in a value of the stress concentration factor very close to the exact one. Furthermore, although the inclusion of more and more coefficients improves the accuracy of the result, this improvement is not quite appreciable. A few more salient points come to light when the two proposed techniques are compared from the standpoint of their application to the nonlinear problem.

To begin with, consider the application of the collocation method to the nonlinear problem. Firstly, the substitution of the assumed solution for φ [Eq. (4.2)], satisfying the boundary conditions (4.1), and Eqs. (4.4a) and (4.5) into the governing equation (3.7) leads to a set of nonlinear algebraic equations in the coefficients A_{k0} and A_{k2}. Secondly, the uncoupling of this set for the separate evaluation of A_{k0} and A_{k2} is not possible. Further, if an iterative scheme is set up for the solution of the set of equations, it must insure the convergence of all the coefficients A_{k0} and A_{k2}. This convergence is extremely difficult to obtain. On the other hand, consider the application of Galerkin's method to the nonlinear problem. This requires the replacement of $\nabla^4\varphi$ in Eqs. (4.8) and (4.9) by the left-hand side of Eq. (3.7). Thus, substitution of the assumed solution for φ, Eq. (4.11a), which satis-

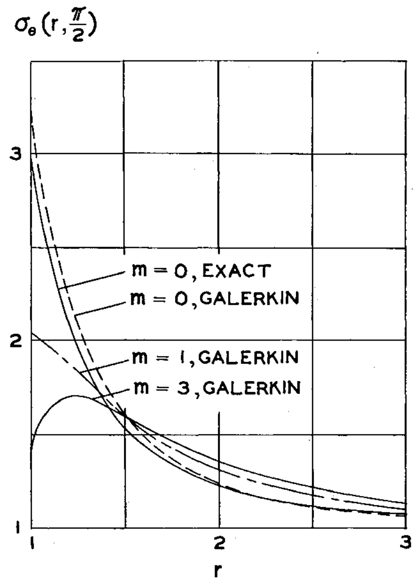

Fig. 2 Circumferential stress for the plate under uniaxial tension at $\theta = \pi/2$.

fies the boundary conditions (4.1), into the Galerkin integral, leads to a set of simultaneous equations for the determination of the coefficients A_{k0} and A_{k2}. At variance with the linear case, a direct evaluation of this integral is exremely difficult. However, the well-known Newton-Raphson technique[5] of successive iteration can be used to determine the coefficients. Moreover, in view of the results obtained in the linear case, only a small number of coefficients A_{k0} and A_{k2} need be considered. Thus, Galerkin's method was used as the more suitable one in the solution of the nonlinear problem.

In obtaining the solution of the nonlinear problem by the proposed technique only one coefficient in each set (A_{20} and A_{22}) was retained. The computations were performed for three values of the creep exponent, $m = 0$, 1, and 3. The results for the stress concentration factor at $r = 1$ and the coefficients A_{20} and A_{22} are summarized in Table 3. Now, the maximum value of the circumferential stress σ_θ occurs

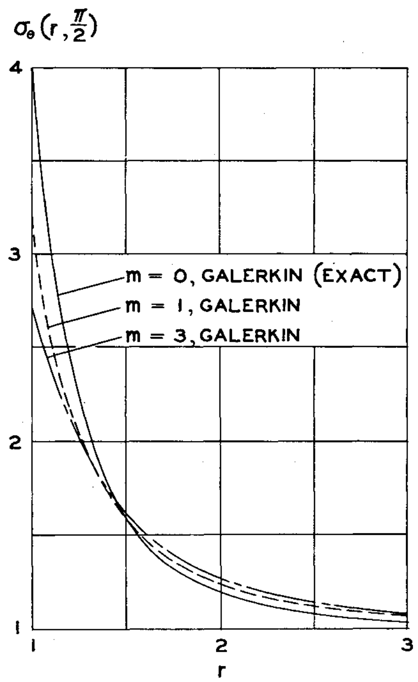

Fig. 3 Circumferential stress for the biaxially loaded plate at $\theta = \pi/2$.

Table 3 Stress concentration factor, A_{20}, and A_{22} for m = 0, 1, and 3

m	0	1	3
Stress conc. factor	3.21678	2.04722	1.42264
A_{20}	−0.14062	−0.28934	−0.12310
A_{22}	−0.250	0.18705	0.66558

at $\theta = \pi/2$ and in the neighborhood of the hole. Away from the hole this normal stress gradually decreases and approaches the value of the applied tension. This stress distribution is shown plotted in Fig. 2. As expected, the stress concentration factor decreases with increasing values of the exponent m. It may be noted that when $m = 3$ the maximum σ_θ does not occur at $r = 1$ but at $r = 1.224$ and has a value 1.703.

5. Biaxially Loaded Plate

Again, if the applied stresses, each of intensity σ_0, and the radius a of the hole (Fig. 1b) are taken as unities, Eq. (3.7) governing the problem can be treated as a dimensionless equation. Therefore, the boundary conditions of the present problem are

$$\sigma_r = 0 \qquad \tau_{r\theta} = 0 \qquad \text{at } r = 1$$
$$\sigma_r = \cos 2\theta \qquad \sigma_\theta = -\cos 2\theta \qquad \tau_{r\theta} = -\sin 2\theta \qquad \text{at } r = \infty \tag{5.1}$$

With a view towards applying Galerkin's procedure to the solution of the problem, it is assumed that the form of function φ is the same as that in Eq. (4.2). Hence, the stresses given by Eqs. (4.3) retain their validity. Further, as in the preceding problem, if the stresses are to remain finite and satisfy the boundary conditions (5.1), it follows that

$$\left.\begin{aligned} & k \geq -2 \qquad l = 2 \\ & \sigma_r = 0 = -\sum_{k=-2}^{\infty} (k+4)A_{k2}\cos 2\theta \\ & \tau_r = 0 = -\sum_{k=-2}^{\infty} 2(k+1)A_{k2}\sin 2\theta \end{aligned}\right\} \tag{5.2a}$$

$$\left.\begin{aligned} \sigma_r &= \cos 2\theta = -2A_{-22}\cos 2\theta \\ \tau_{r\theta} &= -\sin 2\theta = 2A_{-22}\sin 2\theta \\ \sigma_\theta &= -\cos 2\theta = 2A_{-22}\cos 2\theta \end{aligned}\right\} \tag{5.2b}$$

Since Eqs. (5.2b) must hold for all values of θ, the coefficients satisfy the conditions

$$\sum_{k=-2}^{\infty} (k+4)A_{k2} = 0 \qquad \sum_{k=-2}^{\infty} (k+1)A_{k2} = 0$$
$$A_{-22} = -\tfrac{1}{2} \tag{5.3}$$

Now, the application of Galerkin's procedure first requires the substitution of the assumed solution for φ [Eq. (4.2)], which satisfies the boundary conditions (5.1), into the Galerkin integral. If the integral satisfies the condition

$$\int_0^{\pi/2}\int_1^{\infty} D(\varphi)\delta\varphi r dr d\theta = 0 \tag{5.4}$$

Table 4 Stress concentration factor, A_{22}, and A_{32} for m = 0, 1, and 3

m	0	1	3
Stress conc. factor	4	3.17772	2.72043
A_{22}	−0.5	0.02228	−0.22166
A_{32}	0	−0.03705	0.12048

the assumed function will be a solution of Eq. (3.7). Here, $D(\varphi)$ is the expression on the left-hand side of Eq. (3.7). Thus, carrying out the variation $\delta\varphi$ in terms of the coefficients, Eq. (5.4) reduces to

$$\int_0^{\pi/2}\int_1^{\infty} D(\varphi)\frac{\partial\varphi}{\partial A_{l2}}\, rdrd\theta = 0 \tag{5.5}$$

where l takes on, in turn, each and every possible value of k. As in the preceding problem, consideration of a finite number of coefficients A_{k2} would lead to a determinate set of equations for the evaluation of these coefficients. Before proceeding with the aforementioned variations, however, examination of the function φ given by Eq. (4.2) shows that for the Galerkin integral [Eq. (5.5)] to remain finite

$$A_{-12} = 0 \tag{5.6}$$

Thus, the stress function and stresses in Eqs. (4.2) and (4.3) are subject to the conditions in Eqs. (5.3) and (5.6).

Finally, except in the linear case $m = 0$, again a direct evaluation of the Galerkin integral [Eq. (5.5)] is extremely difficult, and, hence, the Newton-Raphson technique is used to determine the coefficients. As before, only two coefficients (A_{22} and A_{32}) were taken into account in the calculation of the stress concentration factor for two values of the creep exponent, $m = 1$ and 3. The results, together with that for the linear case (exact solution), are summarized in Table 4. As in the preceding case, the stress distribution $\sigma_\theta(r, \pi/2)$ is shown plotted in Fig. 3.

In conclusion, a brief mention of a point of interest in the linear case is deemed appropriate. When the preceding procedure is applied to this problem, it turns out that $A_{22} = -\frac{1}{2}$ and all the rest of the coefficients vanish. This leads to the exact concentration factor of 4.

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Axisymmetric Stress Distribution in Anisotropic Cylinders of Finite Length

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The equilibrium and compatibility equations for an axisymmetrically loaded cylinder are reduced to a single integrodifferential equation in the shear stress. The cylinder is assumed to be an elastic orthotropic body possessing cylindrical anisotropy. Arbitrary boundary tractions and a temperature distribution are included. A double collocation procedure utilizing station functions is used to solve the resulting equation in the shear stress. An example of a cylinder subject to both radial and axial temperature variations is solved. A small number of collocation stations is sufficient to produce accurate results.

Nomenclature

a	= inner radius of cylinder
b	= outer radius of cylinder
$d_1, e_1, d_2, e_2, f_1, g_1, f_2, g_2$	= boundary tractions
E_1, E_2, E_3	= Young's moduli
e	= dimensionless free thermal strain
G	= rigidity modulus
l	= cylinder half length
r, z	= radial and axial coordinates
$P_i(\rho), Q_j(\zeta)$	= radial and axial station functions
S_r, S_θ, S_z, S	= dimensionless radial, tangential, axial, and shear stresses
T	= temperature
α	= a/b = dimensionless inner radius
β	= l/b = cylinder length parameter
γ	= anisotropy parameter
$\delta_1, \epsilon_1, \delta_2, \epsilon_2, \varphi_1, \gamma_1, \varphi_2, \gamma_2$	= dimensionless boundary tractions
ϵ	= free thermal strain
$\epsilon_r, \epsilon_\theta, \epsilon_z, \gamma_{rz}$	= radial, tangential, axial, and shear strains
$\nu_{12}, \nu_{13}, \nu_{21}, \nu_{23}, \nu_{31}, \nu_{32}$	= Poisson's ratios
ν_{55}	= fictitious Poisson's ratio
ρ, ζ	= dimensionless radial and axial coordinates
$\sigma_r, \sigma_\theta, \sigma_z, \tau$	= radial, tangential, axial, and shear stresses

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Introduction

WITH an increasing use of aeolotropic materials in practice, the traditional tools of analysis which were extensively developed for application in the case of isotropic materials need to be extended to the anisotropic media.

In the realm of isotropic cylinders, several investigations were performed in the past.[1,2] Mendelson and Roberts[1]